CONTINUOUS HOMOMORPHISMS OF ARENS-MICHAEL ALGEBRAS

ALEX CHIGOGIDZE

ABSTRACT. It is shown (Theorem 3.3) that every continuous homomorphism of Arens-Michael algebras can be obtained as the limit of a morphism of certain projective systems consisting of Fréchet algebras. Based on this we prove (Theorem 4.2) that a complemented subalgebra of an uncountable product of Fréchet algebras is topologically isomorphic to the product of Fréchet algebras. These results are used to characterize (Theorem 5.2) injective objects of the category of locally convex topological vector spaces. Dually, it is shown (Theorem 6.2) that a complemented subspace of an uncountable direct sum of Banach spaces is topologically isomorphic to the direct sum of (**LB**)-spaces. This result is used to characterize (Theorem **??**) projective objects of the above category.

1. INTRODUCTION

Arens-Michael algebras are limits of projective systems of Banach algebras (or, alternatively, closed subalgebras of (uncountable) products of Banach algebras). Quite often, when dealing with a particular Arens-Michael algebra, at least one projective system arises naturally (for instance, as a result of certain construction) and, in most cases, it does contain the needed information about its limit. The situation is somewhat different if an Arens-Michael algebra is given arbitrarily and there is no particular projective system associated with it in a canonical way.

Below (Definition 3.1) we introduce the concept of a projective Fréchet system and show (Theorem 3.3) that every continuous homomorphism of Arens-Michael agebras can be obtained as the limit homomorphism of certain morphism of cofinal subsystems of the corresponding Fréchet systems. This result applied to the identity homomorphism obviously implies that any Arens-Michael algebra has essentially unique Fréchet system associated with it. Consequently, any information about an Arens-Michael algebra is contained in the associated Fréchet system. The remaining problem of restoring this information is, of course, still

non-trivial, but sometimes can be successfully handled by using a simple but effective method. This method is based on Proposition 2.5[1].

Applying such an approach we obtain the following, in a sense dual, statements (precise statements are recorded in Theorems 4.2 and 6.2 respectively).

Retracts of uncountable products of Banach spaces. *A retract of an uncountable product of Banach spaces is the product of retracts of countable subproducts.*

Retracts of uncountable coproducts of Banach spaces. *A retract of an uncountable coproduct of Banach spaces is the coproduct of retracts of countable subcoproducts.*

Based on these results we present a complete description of injective and projective objects of the category $\mathcal{LCS}$ of locally convex topological vector spaces (over the field of complex numbers). Obviously products (coproducts) of arbitrary collections of injective (projective) objects of any category are again injective (projective). Also the class of injectives (projectives) is stable under retractions. Sometimes in a relatively nice category there naturally exists a class of "simple" injectives (projectives) so that any other injective (projective) can be obtained from simple ones by applying the above mentioned operations, i.e. by forming products (coproducts) and by passing to retracts. Consider some examples:

1. Injectives in the category $\mathcal{COMP}$ of compact Hausdorff spaces are precisely retracts of products of copies of the closed unit segment [2], whereas projectives are retracts of coproducts of copies of the singleton [4, Problem 6.3.19(a)]. The cone of the uncountable power of the segment is an example of an injective which is not the product of simpler injectives.
2. Injectives in the category $\mathcal{COMPABGR}$ of compact Hausdorff abelian topological groups are precisely the tori, i.e. products of copies of the circle group $\mathbb{T}$ (see [1] for a topological characterization of arbitrary tori).
3. Projectives in the category $\mathcal{R}$-$\mathcal{MOD}$ of left $\mathcal{R}$-modules (where $\mathcal{R}$ is an associative ring with unit) are coproducts of countably generated projectives [6].

In these examples injectives are retracts of products and projectives are retracts of coproducts of simpler objects. On the other hand, as shows the first example, injectives need not be products.

The following two results (Theorems 5.2 and **??** respectively) provide the full solution of the corresponding problems in the category $\mathcal{LCS}$.

Characterization of injectives. *The following conditions are equivalent for a locally convex topological vector space X:*

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[1]Applications of Proposition 2.5 in a variety of situations can be found in [2].

(1) X is an injective object of the category $\mathcal{LCS}$.

(2) X is isomorphic to the product $\prod\{F_t \colon t \in T\}$, where each F_t, $t \in T$, is a complemented subspace of the product $\prod\{\ell_\infty(J_{t_n}) \colon n \in \omega\}$.

2. Preliminaries

2.1. **Projective systems and their morphisms.** Below we consider projective systems $\mathcal{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$ consisting of topological algebras X_α, $\alpha \in A$, and continuous homomorphisms $p_\alpha^\beta \colon X_\beta \to X_\alpha$, $\alpha \leq \beta$, $\alpha, \beta \in A$ (A is the directed indexing set of $\mathcal{S}_X$). The limit $\varprojlim \mathcal{S}_X$ of this system is defined as the closed subalgebra of the Cartesian product $\prod\{X_\alpha \colon \alpha \in A\}$ (with coordinatewise defined operations) consisting of all *threads* of $\mathcal{S}_X$, i.e.

$$\varprojlim \mathcal{S}_X = \left\{ x_\alpha \in \prod\{X_\alpha \colon \alpha \in A\} \colon p_\alpha^\beta(x_\beta) = x_\alpha \text{ for any } \alpha, \beta \in A \text{ with } \alpha \leq \beta \right\}.$$

The α-th limit projection $p_\alpha \colon \varprojlim \mathcal{S}_X \to X_\alpha$, $\alpha \in A$, of the system $\mathcal{S}_X$ is the restriction (onto $\varprojlim \mathcal{S}_X$) of the α-th natural projection $\pi_\alpha \colon \prod\{X_\alpha \colon \alpha \in A\} \to X_\alpha$.

If A' is a directed subset of the indexing set A, then the subsystem $\{X_\alpha, p_\alpha^\beta, A'\}$ of $\mathcal{S}_X$ is denoted $\mathcal{S}_X | A'$. We refer the reader to [8], [9] for general properties of projective systems.

Suppose we are given two projective systems $\mathcal{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$ and $\mathcal{S}_Y = \{Y_\gamma, q_\gamma^\delta, B\}$ consisting of topological algebras X_α, $\alpha \in A$, and Y_γ, $\gamma \in B$. A *morphism of the system $\mathcal{S}_X$ into the system $\mathcal{S}_Y$* is a family $\{\varphi, \{f_\gamma \colon \gamma \in B\}\}$, consisting of a nondecreasing function $\varphi \colon B \to A$ such that the set $\varphi(B)$ is cofinal in A, and of continuous homomorphisms $f_\gamma \colon X_{\varphi(\gamma)} \to Y_\gamma$ defined for all $\gamma \in B$ such that

$$q_\gamma^\delta f_\delta = f_\gamma p_{\varphi(\gamma)}^{\varphi(\delta)},$$

whenever $\gamma, \delta \in B$ and $\gamma \leq \delta$. In other words, we require (in the above situation) the commutativity of the following diagram

$$
\begin{array}{ccc}
X_{\varphi(\delta)} & \xrightarrow{f_\delta} & Y_\delta \\
{\scriptstyle p_{\varphi(\gamma)}^{\varphi(\delta)}}\Big\downarrow & & \Big\downarrow{\scriptstyle q_\gamma^\delta} \\
X_{\varphi(\gamma)} & \xrightarrow{f_\gamma} & Y_\gamma
\end{array}
$$

Any morphism

$$\{\varphi, \{f_\gamma \colon \gamma \in B\}\} \colon \mathcal{S}_X \to \mathcal{S}_Y$$

induces a continuous homomorphism, called *the limit homomorphism of the morphism*

$$\varprojlim \{\varphi, \{f_\gamma \colon \gamma \in B\}\} \colon \varprojlim \ \mathcal{S}_X \to \varprojlim \ \mathcal{S}_Y.$$

To see this, assign to each thread $x = \{x_\alpha \colon \alpha \in A\}$ of the system $\mathcal{S}_X$ the point $y = \{y_\gamma \colon \gamma \in B\}$ of the product $\prod\{Y_\gamma \colon \gamma \in B\}$ by letting

$$y_\gamma = f_\gamma(x_{\varphi(\gamma)}), \gamma \in B.$$

It is easily seen that the point $y = \{y_\gamma \colon \gamma \in B\}$ is in fact a thread of the system $\mathcal{S}_Y$. Therefore, assigning to $x = \{x_\alpha \colon \alpha \in A\} \in \varprojlim \ \mathcal{S}_X$ the point $y = \{y_\gamma \colon \gamma \in B\} \in \varprojlim \ \mathcal{S}_Y$, we define a map $\varprojlim \{\varphi, \{f_\gamma \colon \gamma \in B\}\} \colon \varprojlim \ \mathcal{S}_X \to \varprojlim \ \mathcal{S}_Y$. Straightforward verification shows that this map is a continuous homomorphism.

Morphisms of projective systems which arise most frequently in practice are those defined over the same indexing set. In this case, the map $\varphi \colon A \to A$ of the definition of morphism is taken to be the identity. Below we shall mostly deal with such situations and use the following notation: $\{f_\alpha \colon X_\alpha \to Y_\alpha; \alpha \in A\} \colon \mathcal{S}_X \to \mathcal{S}_Y$ or sometimes even a shorter form $\{f_\alpha\} \colon \mathcal{S}_X \to \mathcal{S}_Y$.

Proposition 2.1. *Let $\mathcal{S}_Y = \{Y_\alpha, q_\alpha^\beta, A\}$ be a projective system and X be a topological algebra. Suppose that for each $\alpha \in A$ a continuous homomorphism $f_\alpha \colon X \to Y_\alpha$ is given in such a way that $f_\alpha = q_\alpha^\beta f_\beta$ whenever $\alpha, \beta \in A$ and $\alpha \leq \beta$. Then there exists a natural continuous homomorphism $f \colon X \to \varprojlim \ \mathcal{S}_Y$ (the diagonal product $\triangle\{f_\alpha \colon \alpha \in A\}$) satisfying, for each $\alpha \in A$, the condition $f_\alpha = q_\alpha f$.*

Proof. Indeed, we only have to note that X, together with its identity map id_X, forms the projective system $\mathcal{S}$. So the collection $\{f_\alpha \colon \alpha \in A\}$ is in fact a morphism $\mathcal{S} \to \mathcal{S}_Y$. The rest follows from the definitions given above. $\qquad\square$

2.2. **Arens-Michael algebras.** We recall some definitions [5]. A *polynormed space* X is a topological linear space X furnished with a collection $\{||\cdot||_\nu, \Lambda\}$ of seminorms generating the topology of X. This simply means that the collection

$$\{x \in X \colon ||x - x_0||_\nu < \epsilon\}, x_0 \in X, \nu \in \Lambda, \epsilon > 0,$$

forms a subbase of the topology of X. A *polynormed algebra* is a polynormed space X wich admits a separately continuous multiplication bioperator $m \colon X \times X \to X$. A *multinormed algebra* is a polynormed algebra such that $||xy||_\nu \leq ||x||_\nu \cdot ||y||_\nu$ for each $\nu \in \Lambda$ and any $(x, y) \in X \times X$. Finally, an *Arens-Michael algebra* is a complete (and Hausdorff) multinormed algebra.

The following result [5, Corollary V.2.19] provides a characterization of Arens-Michael algebras.

Theorem 2.2. *The following conditions are equivalent for a multinormed algebra X:*

(a) *X is an Arens-Michael algebra.*
(b) *X is limit of a certain projective system of Banach algebras.*
(c) *X is topologically isomorphic to a closed subalgebra of the Cartesian product of a certain family of Banach algebras.*

2.3. **Set-theoretical facts.** For the reader's convenience we begin by presenting necessary set-theoretic facts. Their complete proofs can be found in [2].

Let A be a partially ordered *directed set* (i.e. for every two elements $\alpha, \beta \in A$ there exists an element $\gamma \in A$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$). We say that a subset $A_1 \subseteq A$ of A *majorates* another subset $A_2 \subseteq A$ of A if for each element $\alpha_2 \in A_2$ there exists an element $\alpha_1 \in A_1$ such that $\alpha_1 \geq \alpha_2$. A subset which majorates A is called *cofinal* in A. A subset of A is said to be a *chain* if every two elements of it are comparable. The symbol $\sup B$, where $B \subseteq A$, denotes the lower upper bound of B (if such an element exists in A). Let now τ be an infinite cardinal number. A subset B of A is said to be τ-*closed* in A if for each chain $C \subseteq B$, with $|C| \leq \tau$, we have $\sup C \in B$, whenever the element $\sup C$ exists in A. Finally, a directed set A is said to be τ-*complete* if for each chain C of elements of A with $|C| \leq \tau$, there exists an element $\sup C$ in A.

The standard example of a τ-complete set can be obtained as follows. For an arbitrary set A let $\exp A$ denote, as usual, the collection of all subsets of A. There is a natural partial order on $\exp A$: $A_1 \geq A_2$ if and only if $A_1 \supseteq A_2$. With this partial order $\exp A$ becomes a directed set. If we consider only those subsets of the set A which have cardinality $\leq \tau$, then the corresponding subcollection of $\exp A$, denoted by $\exp_\tau A$, serves as a basic example of a τ-complete set.

Proposition 2.3. *Let $\{A_t : t \in T\}$ be a collection of τ-closed and cofinal subsets of a τ-complete set A. If $| T | \leq \tau$, then the intersection $\cap\{A_t : t \in T\}$ is also cofinal (in particular, non-empty) and τ-closed in A .*

Corollary 2.4. *For each subset B, with $| B | \leq \tau$, of a τ-complete set A there exists an element $\gamma \in A$ such that $\gamma \geq \beta$ for each $\beta \in B$.*

Proposition 2.5. *Let A be a τ-complete set, $L \subseteq A^2$, and suppose that the following three conditions are satisfied:*

    Existence: *For each $\alpha \in A$ there exists $\beta \in A$ such that $(\alpha, \beta) \in L$.*
    Majorantness: *If $(\alpha, \beta) \in L$ and $\gamma \geq \beta$, then $(\alpha, \gamma) \in L$.*
    τ-**closeness:** *Let $\{\alpha_t : t \in T\}$ be a chain in A with $| T | \leq \tau$. If $(\alpha_t, \gamma) \in L$ for some $\gamma \in A$ and each $t \in T$, then $(\alpha, \gamma) \in L$ where $\alpha = \sup\{\alpha_t : t \in T\}$.*

Then the set of all L-reflexive elements of A (an element $\alpha \in A$ is L-reflexive if $(\alpha, \alpha) \in L$) is cofinal and τ-closed in A.

3. Continuous homomorphisms of Arens-Michael algebras

The following statement is needed in the proof of Theorem 3.3. In the case when all X_α's are Banach algebras its proof can be extracted from [5, Proposition 0.1.9] (see also [5, Proof of Proposition V.1.8]).

Lemma 3.1. *Let $\mathcal{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$ be a projective system consisting of Arens-Micheael algebras X_α, $\alpha \in A$, Y be a Banach algebra and $f \colon \varprojlim \mathcal{S} \to Y$ be a continuous homomorphism. Then there exist an index $\alpha \in A$ and a continuous homomorphism $f_\alpha \colon X_\alpha \to Y$ such that $f = f_\alpha \circ p_\alpha$.*

Proof. The continuity of f and the definition of the topology on $\varprojlim \mathcal{S}$ guarantee that there exists an index $\alpha \in A$ and an open subset $V_\alpha \subseteq X_\alpha$ such that

$$(3.1) \qquad f\left(p_\alpha^{-1}(V_\alpha)\right) \subseteq \{y \in Y \colon |y| \leq 1\},$$

where $|\cdot|$ denotes the norm of the Banach space Y.

Since X_α is an Arens-Michael algebra, X_α can be identified with a closed subalgebra of the product $\prod\{B_t \colon t \in T\}$ of Banach algebras B_t, $t \in T$ (Theorem 2.2). Let $||\cdot||_t$ denote the norm of the Banach space B_t, $t \in T$. For $S \subseteq T$, let $\pi_S \colon \prod\{B_t \colon t \in T\} \to \prod\{B_t \colon t \in S\}$ denote the natural projection onto the corresponding subproduct. If $S \subseteq T$ is a finite subset of T, then $||\{x_t \colon t \in S\}||_S = \max\{||x_t||_t \colon t \in S\}$ for each $\{x_t \colon t \in S\} \in \prod\{B_t \colon t \in S\}$.

Since V_α is open in X_α, the definition of the product topology guarantees the existence of a finite subset $S \subseteq T$ and of a number $\epsilon > 0$ such that

$$(3.2) \qquad \{x_\alpha \in X_\alpha \colon ||\pi_S(x_\alpha)||_S \leq \epsilon\} \subseteq V_\alpha$$

Combining (3.1) and (3.2), we have

$$(3.3) \qquad f\left(\{x \in X \colon ||\pi_S\left(p_\alpha(x)\right)||_S \leq \epsilon\}\right) \subseteq \{y \in Y \colon |y| \leq 1\}.$$

It then follows that if $x \in X$ and $||\pi_S(p_\alpha(x))||_S \leq 1$, then $||\pi_S(p_\alpha(\epsilon x))||_S = \epsilon||\pi_S(p_\alpha(x))||_S \leq \epsilon$ and consequently

$$(3.4) \qquad \epsilon|f(x)| = |f(\epsilon x)| \leq 1, \text{ i.e. } |f(x)| \leq \frac{1}{\epsilon}.$$

Since

$$\left\|\pi_S\left(p_\alpha\left(\frac{x}{||\pi_S(p_\alpha(x))||_S}\right)\right)\right\| = \frac{1}{||\pi_S(p_\alpha(x))||_S} \cdot ||\pi_S(p_\alpha(x))||_S = 1,$$

we must have (by 3.4)

$$\frac{1}{||\pi_S(p_\alpha(x))||_S}|f(x)| = \left|f\left(\frac{x}{||\pi_S(p_\alpha(x))||_S}\right)\right| \leq \frac{1}{\epsilon}$$

and hence

(3.5) $$|f(x)| \leq \frac{1}{\epsilon}||\pi_S(p_\alpha(x))||_S \text{ for each } x \in X.$$

Let us now show that the map

$$h(z) = f\left(p_\alpha^{-1}\left(\pi_S^{-1}(z) \cap X_\alpha\right)\right): \pi_S(p_\alpha(X)) \to Y$$

is well defined. Assuming the contrary, suppose that for some $z \in \pi_S(p_\alpha(X))$ there exist two points $x_1, x_2 \in p_\alpha^{-1}\left(\pi_S^{-1}(z) \cap X_\alpha\right)$ such that $f(x_1) \neq f(x_2)$. Consequently, $|f(x_1 - x_2)| \neq 0$. On the other hand, $\pi_S(p_\alpha(x_1 - x_2)) = \pi_S(p_\alpha(x_1)) - \pi_S(p_\alpha(x_2)) = z - z = 0$. Then (3.5) implies that

$$0 \neq |f(x_1 - x_2)| \leq \frac{1}{\epsilon}||\pi_S(p_\alpha(x_1 - x_2))||_S = 0.$$

This contradiction shows that the map $h \colon \pi_S(p_\alpha(X)) \to Y$ is indeed well defined. Note that

(3.6) $$f = h \circ \pi_S \circ p_\alpha,$$

which implies that the map h is linear. Next consider points $z \in \pi_S(p_\alpha(X))$ and $x \in X$ such that $\pi_S(p_\alpha(x)) = z$. By (3.5),

$$|h(z)| = |f(x)| \leq \frac{1}{\epsilon}||\pi_S(p_\alpha(x))||_S = \frac{1}{\epsilon}||z||_S.$$

This shows that h is bounded and, consequently, continuous. Next let us show that $h \colon \pi_S(p_\alpha(X)) \to Y$ is multiplicative. Let $(x', y') \in \pi_S(p_\alpha(X)) \times \pi_S(p_\alpha(X))$ and consider a point $(x, y) \in X \times X$ such that $\pi_S(p_\alpha(x)) = x'$ and $\pi_S(p_\alpha(y)) = y'$. Then, by (3.6),

$$h(x' \cdot y') = h\left(\pi_S(p_\alpha(x)) \cdot \pi_S(p_\alpha(y))\right) = h\left(\pi_S(p_\alpha(x \cdot y))\right) = f(x \cdot y) =$$
$$f(x) \cdot f(y) = h\left(\pi_S(p_\alpha(x))\right) \cdot h\left(\pi_S(p_\alpha(y))\right) = h(x') \cdot h(y').$$

Since $(Y, |\cdot|)$ is complete, h admits the linear continuous extension

$$g \colon \text{cl}_{\prod\{B_t \colon t \in S\}}\left(\pi_S(p_\alpha(X))\right) \to Y.$$

Since the multiplication on $\prod\{B_t \colon t \in T_f\}$ is jointly continuous, we conclude that g is also multiplicative. Finally, define the map f_α as the composition

$$f_\alpha = g \circ (\pi_S|X_\alpha) \colon X_\alpha \to Y.$$

Obviously f_α is a continuous homomorphism satisfying the required equality $f_\alpha \circ p_\alpha = f$. $\square$

Next we introduce the concept of *projective Fréchet system.*

Definition 3.1. Let $\tau \geq \omega$ be a cardinal number. A projective system $\mathcal{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$ consisting of topological algebras X_α and continuous homomorphisms $p_\alpha^\beta \colon X_\beta \to X_\alpha$, $\alpha \leq \beta$, $\alpha, \beta \in A$, is a τ-*system if:*

1. X_α is a closed subalgebra of the product of at most τ Banach algebras, $\alpha \in A$.
2. The indexing set A is τ-complete.
3. If $\{\alpha_\gamma \colon \gamma \in \tau\}$ is an increasing chain of elements in A with $\alpha = \sup\{\alpha_\gamma \colon \gamma \in \tau\}$, then the diagonal product[2]

$$\triangle\{p_{\alpha_\gamma}^\alpha \colon \gamma \in \tau\} \colon X_\alpha \to \varprojlim\{X_{\alpha_\gamma}, p_{\alpha_\gamma}^{\alpha_{\gamma+1}}, \tau\}$$

   is a topological isomorphism.
4. $p_\alpha(X)$ is dense in X_α for each $\alpha \in A$.

Fréchet systems are defined as projective ω-systems.

Proposition 3.2. *Every Arens-Micheal algebra X can be represented as the limit of a projective Fréchet system $\mathcal{S}_X = \{X_A, p_A^B, \exp_\omega T\}$. Conversely, the limit of a projective Fréchet system is an Arens-Michael algebra.*

Proof. By Theorem 2.2, X can be identified with a closed subalgebra of the product $\prod\{X_t \colon t \in T\}$ of some collection of Banach algebras. If $|T| \leq \omega$, then X itself is a Fréchet algebra and therefore our statement is trivially true. If $|T| > \omega$, then consider the set $\exp_\omega T$ of all countable subsets of T. Clearly, $\exp_\omega T$ is ω-complete set (see Subsection 2.3). For each $A \in \exp_\omega T$, let $X_A = \operatorname{cl}\pi_A(X)$ (closure is taken in $\prod\{X_t \colon t \in A\}$), where

$$\pi_A \colon \prod\{X_t \colon t \in T\} \to \prod\{X_t \colon t \in A\}$$

denotes the natural projection onto the corresponding subproduct. Also let $p_A^B = \pi_A^B | X_B$ where

$$\pi_A^B \colon \prod\{X_t \colon t \in B\} \to \prod\{X_t \colon t \in A\}$$

is the natural projection, $A, B \in \exp_\omega T$, $A \leq B$. The straightforward verification shows that $\mathcal{S}_X = \{X_A, p_A^B, \exp_\omega T\}$ is indeed a projective Fréchet system such that $\varprojlim \mathcal{S}_X = X$.

Conversely, let $\mathcal{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$ be a projective Fréchet system. Clearly, $\varprojlim \mathcal{S}_X$ can be identified with a closed subalgebra of the product $\prod\{X_\alpha \colon \alpha \in A\}$ (see Subsection 2.1). Each X_α, $\alpha \in A$, can obviously be identified with a closed subalgebra of the product $\prod\{B_n^\alpha \colon n \in T_\alpha\}$ of a countable collection of Banach algebras B_n^α. Then $\varprojlim \mathcal{S}_X$, as a closed subalgebra of the product $\prod\left\{\prod\{B_n^\alpha \colon n \in T_\alpha\} \colon a \in A\right\}$ is, according to Theorem 2.2, an Arens-Michael algebra. $\square$

The following statement is one of our main results.

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[2]See Proposition 2.1.

Theorem 3.3. *Let* $f\colon \varprojlim \mathcal{S}_X \to \varprojlim \mathcal{S}_Y$ *be a continuous homomorphism between the limits of two projective Fréchet systems* $\mathcal{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$ *and* $\mathcal{S}_Y = \{Y_\alpha, q_\alpha^\beta, A\}$ *with the same indexing set* A. *Then there exist a cofinal and* ω-*closed subset* B_f *of* A *and a morphism*

$$\{f_\alpha\colon X_\alpha \to Y_\alpha, B_f\}\colon \mathcal{S}_X|B_f \to \mathcal{S}_Y|B_f,$$

consisting of continuous homomorphisms $f_\alpha\colon X_\alpha \to Y_\alpha$, $\alpha \in B_f$, *such that* $f = \varprojlim\{f_\alpha\colon B_f\}$.

If, in particular, $\varprojlim \mathcal{S}_X$ *and* $\varprojlim \mathcal{S}_Y$ *are topologically isomorphic, then* $\mathcal{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$ *and* $\mathcal{S}_Y = \{Y_\alpha, q_\alpha^\beta, A\}$ *contain isomorphic cofinal and* ω-*closed subsystems.*

Proof. We perform the spectral search by means of the following relation

$$L = \{(\alpha, \beta) \in A^2\colon \alpha \leq \beta \text{ and there exists a continuous homomorphism}$$
$$f_\alpha^\beta\colon X_\beta \to Y_\alpha \text{ such that } f_\alpha^\beta p_\beta = q_\alpha f\}.$$

Let us verify the conditions of Proposition 2.5.

Existence Condition. By assumption, Y_α is a Fréchet algebra. Therefore Y_α can be identified with a closed subspace of a countable product $\prod\{B_n\colon n \in \omega\}$ of Banach algebras. Let $\pi_n\colon \prod\{B_n\colon n \in \omega\} \to B_n$ denote the n-th natural projection. For each $n \in \omega$, by Lemma 3.1, there exist an index $\beta_n \in A$ and a continuous homomorphism $f_{\beta_n}\colon X_{\beta_n} \to B_n$ such that $\pi_n q_\alpha f = f_{\beta_n} p_{\beta_n}$. By Corollary 2.4, there exists an index $\beta \in A$ such that $\beta \geq \beta_n$ for each n. Without loss of generality we may assume that $\beta \geq \alpha$. Let $f_n = f_{\beta_n} p_{\beta_n}^\beta$, $n \in \omega$. Next consider the diagonal product

$$f_\alpha^\beta = \triangle\{f_n\colon n \in \omega\}\colon X_\beta \to \prod\{B_n\colon n \in \omega\}.$$

Obviously, $f_\alpha^\beta p_\beta = q_\alpha f$. It only remains to show that $f_\alpha^\beta(X_\beta) \subseteq Y_\alpha$. First observe that $f_\alpha^\beta(p_\beta(X)) \subseteq Y_\alpha$. Indeed, let $x \in X$. Then

$$f_\alpha^\beta(p_\beta(x)) = \{f_n(p_\beta(x))\colon n \in \omega\} = \{f_{\beta_n}(p_{\beta_n}^\beta(p_\beta(x)))\colon n \in \omega\} =$$
$$\{f_{\beta_n}(p_{\beta_n}(x))\colon n \in \omega\} = \{\pi_n(q_\alpha(f(x)))\colon n \in \omega\} = q_\alpha(f(x)) \in Y_\alpha.$$

Finally,

$$f_\alpha^\beta(X_\beta) = f_\alpha^\beta(\mathrm{cl}_{X_\beta}\, p_\beta(X)) \subseteq \mathrm{cl}_{\prod\{B_n\colon n\in\omega\}} f_\alpha^\beta(p_\beta(X)) \subseteq \mathrm{cl}_{\prod\{B_n\colon n\in\omega\}} Y_\alpha = Y_\alpha.$$

Majorantness Condition. The verification of this condition is trivial. Indeed, it suffiecs to consider the composition $f_\alpha^\gamma = f_\alpha^\beta p_\beta^\gamma$.

ω-**closeness Condition.** Suppose that for some countable chain $C = \{\alpha_n\colon n \in \omega\}$ in A with $\alpha = \sup C$ and for some $\beta \in A$ with $\beta \geq \alpha$, the maps $f_{\alpha_n}^\beta\colon X_\beta \to$

Y_{α_n} have already been defined is such a way that $f^{\beta}_{\alpha_n} p_{\beta} = q_{\alpha_n} f$ for each $n \in \omega$ (in other words, $(\alpha_n, \beta) \in L$ for each $n \in \omega$). Next consider the composition

$$f^{\beta}_{\alpha} = i^{-1} \circ \triangle\{f^{\beta}_{\alpha_n} : n \in \omega\} : X_{\beta} \to Y_{\alpha},$$

where $i\colon Y_{\alpha} \to \varprojlim\{Y_{\alpha_n}, q^{\alpha_{n+1}}_{\alpha_n}, \omega\}$ is the topological isomorphism indicated in condition 3 of Definition 3.1. Observe that for each $x \in X$

$$f^{\beta}_{\alpha}(p_{\beta}(x)) = i^{-1}(\triangle\{f^{\beta}_{\alpha_n} : n \in \omega\})(p_{\beta}(x)) =$$
$$i^{-1}(\triangle\{q_{\alpha_n} : n \in \omega\})(f(x)) = i^{-1}(\triangle\{q^{\alpha}_{\alpha_n} : n \in \omega\})(q_{\alpha}(f(x))) =$$
$$(\triangle\{q^{\alpha}_{\alpha_n} : n \in \omega\})^{-1}(\triangle\{q^{\alpha}_{\alpha_n} : n \in \omega\})(q_{\alpha}(f(x))) = q_{\alpha}(f(x)).$$

This shows that $(\alpha, \beta) \in L$ and finishes the verification of the ω-closeness condition.

Now denote by B_f the set of all L-reflexive elements in A. By Proposition 2.5, B_f is a cofinal and ω-closed subset of A. One can easily see that the L-reflexivity of an element $\alpha \in A$ is equivalent to the existence of a continuous homomorphism $f_{\alpha} = f^{\alpha}_{\alpha} \colon X_{\alpha} \to Y_{\alpha}$ satisfying the equality $f_{\alpha} p_{\alpha} = q_{\alpha} f$. Consequently, the collection $\{f_{\alpha} \colon \alpha \in B_f\}$ is a morphism of the cofinal and ω-closed subspectrum $\mathcal{S}_X | B_f$ of the spectrum $\mathcal{S}_X$ into the cofinal and ω-closed subspectrum $\mathcal{S}_Y | B_f$ of the spectrum $\mathcal{S}_Y$. It only remains to remark that the original map f is induced by the constructed morphism. This finishes the proof of the first part of our Theorem.

The second part of this theorem can be obtained from the first as follows. Let $f\colon \varprojlim \mathcal{S}_X \to \varprojlim \mathcal{S}_Y$ be a topological isomorphism. Denote by $f^{-1}\colon \varprojlim \mathcal{S}_Y \to \varprojlim \mathcal{S}_X$ the inverse of f. By the first part proved above, there exist a cofinal and ω-closed subset B_f of A and a morphism

$$\{f_{\alpha} \colon X_{\alpha} \to Y_{\alpha} \colon \alpha \in B_f\} \colon \mathcal{S}_X | B_f \to \mathcal{S}_Y | B_f$$

such that $f = \varprojlim\{f_{\alpha} \colon \alpha \in B_f\}$. Similarly, there exist a cofinal and ω-closed subset $B_{f^{-1}}$ of A and a morphism

$$\{g_{\alpha} \colon Y_{\alpha} \to X_{\alpha} \colon \alpha \in B_{f^{-1}}\} \colon \mathcal{S}_Y | B_{f^{-1}} \to \mathcal{S}_X | B_{f^{-1}}$$

such that $f^{-1} = \varprojlim\{g_{\alpha} \colon \alpha \in B_{f^{-1}}\}$.

By Proposition 2.3, the set $B = B_f \cap B_{f^{-1}}$ is still cofinal and ω-closed in A. Therefore, in order to complete the proof, it suffices to show that for each $\alpha \in B$ the map $f_{\alpha} \colon X_{\alpha} \to Y_{\alpha}$ is a topological isomorphism. Indeed, take a point $x_{\alpha} \in p_{\alpha}\left(\varprojlim \mathcal{S}_X\right) \subseteq X_{\alpha}$. Also choose a point $x \in \varprojlim \mathcal{S}_X$ such that $x_{\alpha} = p_{\alpha}(x)$. Then

$$g_{\alpha} f_{\alpha}(x_{\alpha}) = g_{\alpha} f_{\alpha} p_{\alpha}(x) = g_{\alpha} q_{\alpha} f(x) = p_{\alpha} f^{-1} f(x) = p_{\alpha}(x) = x_{\alpha}.$$

This proves that $g_\alpha f_\alpha | p_\alpha \left(\varprojlim \mathcal{S}_X \right) = \mathrm{id}_{p_\alpha \left(\varprojlim \mathcal{S}_X \right)}$. Similar considerations show that $f_\alpha g_\alpha | q_\alpha \left(\varprojlim \mathcal{S}_Y \right) = \mathrm{id}_{q_\alpha \left(\varprojlim \mathcal{S}_Y \right)}$ for each $\alpha \in B$. Since $p_\alpha \left(\varprojlim \mathcal{S}_X \right)$ is dense in X_α and $q_\alpha \left(\varprojlim \mathcal{S}_Y \right)$ is dense in Y_α (condition 4 of Definition 3.1), it follows that $g_\alpha f_\alpha | X_\alpha = \mathrm{id}_{X_\alpha}$ and $f_\alpha g_\alpha | Y_\alpha = \mathrm{id}_{Y_\alpha}$. It is now clear that f_α, $\alpha \in B$, is a topological isomorphism (whose inverse is g_α). $\square$

Remark 3.1. A similar statement remains true (with the identical proof) for projective τ-systems for any cardinal number $\tau > \omega$.

Remark 3.2. Theorem 3.3 is false for countable projective systems. Indeed, consider the following two projective sequences

$$\mathcal{S}_{\mathrm{even}} = \{\mathbb{C}^{2n}, \pi_{2n}^{2(n+1)}, \omega\} \text{ and } \mathcal{S}_{\mathrm{odd}} = \{\mathbb{C}^{2n+1}, \pi_{2n+1}^{2(n+1)+1}, \omega\},$$

where

$$\pi_{2n}^{2(n+1)} \colon \mathbb{C}^{2n} \times \mathbb{C}^2 \to \mathbb{C}^{2n} \text{ and } \pi_{2n+1}^{2(n+1)+1} \colon \mathbb{C}^{2n+1} \times \mathbb{C}^2 \to \mathbb{C}^{2n+1}$$

denote the natural projections. Clearly the limits $\varprojlim \mathcal{S}_{\mathrm{even}}$ and $\varprojlim \mathcal{S}_{\mathrm{odd}}$ of these projective systems are topologically isomorphic (both are topologically isomorphic to the countable infinite power $\mathbb{C}^\omega$ of $\mathbb{C}$), but $\mathcal{S}_{\mathrm{even}}$ and $\mathcal{S}_{\mathrm{odd}}$ do not contain isomorphic cofinal subsystems.

Corollary 3.4. *Let $\mathcal{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$ be a projective Fréchet system. If $\varprojlim \mathcal{S}$ is a Fréchet algebra, then there exists an index $\alpha \in A$ such that the β-th limit projection $p_\beta \colon \varprojlim \mathcal{S} \to X_\beta$ is a topological isomorphism for each $\beta \geq \alpha$.*

Proof. Consider a trivial projective Fréchet system $\mathcal{S}' = \{X_\alpha, q_\alpha^\beta, A\}$, where $X_\alpha = \varprojlim \mathcal{S}$ and $q_\alpha^\beta = \mathrm{id}_{\varprojlim \mathcal{S}}$ for each $\alpha, \beta \in A$. By Theorem 3.3 (applied to the identity homomorphism $f = \mathrm{id}_{\varprojlim \mathcal{S}}$), there exist an index $\alpha \in A$ and a continuous homomorphism $g_\alpha \colon X_\alpha \to \varprojlim \mathcal{S}$ such that $\mathrm{id}_{\varprojlim \mathcal{S}} = g_\alpha \circ p_\alpha$. Clearly, in this situation, $p_\alpha | \varprojlim \mathcal{S} \colon \varprojlim \mathcal{S} \to X_\alpha$ is an embedding with a closed image. But this image $p_\alpha(\varprojlim \mathcal{S})$ is dense in X_α (condition (4) of Definition 3.1). Therefore p_α (and, consequently p_β for any $\beta \geq \alpha$) is a topological isomorphism. $\square$

Corollary 3.5. *Suppose that X is a Fréchet subalgebra of an uncountable product $\prod\{B_t \colon t \in T\}$ of Fréchet (Banach) algebras, then there exists a countable subset T_X of the indexing set T such that the restriction $\pi_{T_X} | X \colon X \to \pi_{T_X}(X)$ of the natural projection $\pi_{T_X} \colon \prod\{B_t \colon t \in T\} \to \prod\{B_t \colon t \in T_X\}$ is a topological isomorphism.*

3.1. **Arens-Michael $*$-algebras.** The concept of projective Fréchet system can naturally be adjusted to handle variety of situations. Below it will always be completely clear in what content this concept is being used. Let us consider Arens-Michael $*$-algebras, i.e. Arens-Michael algebras with a continuous

involution. It is known that every such an algebra can be identified with a closed $*$-subalgebra of the product of Banach $*$-algebras (see, for instance, [5, Proposition V.3.41]). Therefore one can obtain an alternative description of such algebras as limits of projective systems consisting of Banach $*$-algebras and continuous $*$-homomorphisms (compare with Theorem 2.2). This, as in Proposition 3.2, leads us to the conclusion recorded in the following statement.

Proposition 3.6. *Every Arens-Micheal $*$-algebra X can be represented as the limit of a projective Fréchet system $\mathcal{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$ consisting of Fréchet $*$-algebras X_α, $\alpha \in A$, and continuous $*$-homomorphisms $p_\alpha^\beta\colon X_\beta \to X_\alpha$, $\alpha \leq \beta$, $\alpha, \beta \in A$. Conversely, the limit of any such projective Fréchet system is an Arens-Michael $*$-algebra.*

The analog of Theorem 3.3 is also true.

Proposition 3.7. *Let $f\colon \varprojlim \mathcal{S}_X \to \varprojlim \mathcal{S}_Y$ be a continuous $*$-homomorphism between the limits of two projective Fréchet systems $\mathcal{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$ and $\mathcal{S}_Y = \{Y_\alpha, q_\alpha^\beta, A\}$, consisting of Fréchet $*$-algebras and continuous $*$-homomorphisms and having the same indexing set A. Then there exist a cofinal and ω-closed subset B_f of A and a morphism*

$$\{f_\alpha\colon X_\alpha \to Y_\alpha, B_f\}\colon \mathcal{S}_X|B_f \to \mathcal{S}_Y|B_f,$$

consisting of continuous $*$-homomorphisms $f_\alpha\colon X_\alpha \to Y_\alpha$, $\alpha \in B_f$, such that $f = \varprojlim\{f_\alpha; B_f\}$.

If, in particular, $\varprojlim \mathcal{S}_X$ and $\varprojlim \mathcal{S}_Y$ are topologically $*$-isomorphic, then $\mathcal{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$ and $\mathcal{S}_Y = \{Y_\alpha, q_\alpha^\beta, A\}$ contain isomorphic cofinal and ω-closed subsystems.

Proof. By Theorem 3.3, there exists a cofinal and ω-closed subset B_f of A and a morphism

$$\{f_\alpha\colon X_\alpha \to Y_\alpha, B_f\}\colon \mathcal{S}_X|B_f \to \mathcal{S}_Y|B_f,$$

consisting of continuous homomorphisms $f_\alpha\colon X_\alpha \to Y_\alpha$, $\alpha \in B_f$, such that $f = \varprojlim\{f_\alpha; B_f\}$.

Let us show that f_α, $\alpha \in B_f$ is actually a $*$-homomorphism. Indeed, let $x_\alpha \in p_\alpha(X)$ and $x \in X$ such that $p_\alpha(x) = x_\alpha$. Then

$$f_\alpha(x_\alpha^*) = f_\alpha\left(p_\alpha(x)^*\right) = f_\alpha(p_\alpha(x^*)) = q_\alpha(f(x^*)) = q_\alpha\left(f(x)^*\right) =$$
$$q_\alpha(f(x))^* = f_\alpha(p_\alpha(x))^* = f_\alpha(x_\alpha)^*.$$

$\square$

4. Complemented subalgebras of uncountable products of Fréchet algebras

In this section we show (Theorem 4.2) that complemented subalgebras of (uncountable) products of Fréchet algebras are products of Fréchet algebras. We begin with the following lemma.

Lemma 4.1. *Let $p\colon X \to Y$ be a surjective continuous homomorphism of topological algebras and suppose that X is a closed subalgebra of the product $Y \times B$, where B is a a topological algebra. Assume also that there exists a continuous homomorphism $r\colon Y \times B \to X$ satisfying the following conditions:*

   (i) *$pr = \pi_Y$, where $\pi_Y\colon Y \times B \to Y$ denotes the natural projection;*
   (ii) *$r(x) = x$ for each $x \in X$.*

Then there exists a topological isomorphism $h\colon X \to Y \times \ker p$ such that $\pi_Y h = p$.

Proof. If $x \in X$, then

$$p\left(x - r(p(x),0)\right) = p(x) - p(r(p(x),0)) \stackrel{\mathrm{by}\,(i)}{=}$$
$$p(x) - \pi_Y(p(x),0) = p(x) - p(x) = 0.$$

This shows that the formula

$$h(x) = \left(p(x), x - r(p(x),0)\right), x \in X,$$

defines a continuous linear map $h\colon X \to Y \times \ker p$. Moreover, h is a topological isomorphism between X and $Y \times \ker p$ considered as topological vector spaces (to see this observe that the continuous and linear map $g\colon Y \times \ker p \to X$ defined by letting $g(y,x) = r(y,0) + x$ for each $(y,x) \in Y \times \ker p$, has the following properties: $g \circ h = \mathrm{id}_X$ and $h \circ g = \mathrm{id}_{Y \times \ker p}$). We now show that h is an isomorophism of the category of topological algebras as well.

Let $x_1, x_2 \in X$. We need to show that $h(x_1) \cdot h(x_2) = h(x_1 \cdot x_2)$. Since $X \subseteq Y \times B$ we can write $x_i = (a_i, b_i)$, where $a_i \in Y$ and $b_i \in B$, $i = 1,2$. Observe that since $x_i \in X$ it follows from (ii) that $r(x_i) = x_i$. Consequently, by (i), $p(x_i) = p(r(x_i)) = \pi_Y(x_i) = \pi_Y(a_i, b_i) = a_i$. Then

$$h(x_i) = (p(x_i), x_i - r(p(x_i),0)) = (a_i, (a_i, b_i) - r(a_i,0)), i = 1,2.$$

Consequently,

$$
\begin{aligned}
h(x_1) \cdot h(x_2) &= (a_1, (a_1, b_1) - r(a_1, 0)) \cdot (a_2, (a_2, b_2) - r(a_2, 0)) = \\
&\quad (a_1 \cdot a_2, [(a_1, b_1) - r(a_1, 0)] \cdot [(a_2, b_2) - r(a_2, 0)]) = \\
&\quad (a_1 \cdot a_2, [r(a_1, b_1) - r(a_1, 0)] \cdot [r(a_2, b_2) - r(a_2, 0)]) = \\
&\quad (a_1 \cdot a_2, r(0, b_1) \cdot r(0, b_2)) = (a_1 \cdot a_2, r[(0, b_1) \cdot (0, b_2)]) = \\
&\quad (a_1 \cdot a_2, r(0, b_1 \cdot b_2)) = (a_1 \cdot a_2, r[(a_1 \cdot a_2, b_1 \cdot b_2) - (a_1 \cdot a_2, 0)]) = \\
&\quad (a_1 \cdot a_2, r(a_1 \cdot a_2, b_1 \cdot b_2) - r(a_1 \cdot a_2, 0)) = \\
&\quad (a_1 \cdot a_2, (a_1 \cdot a_2, b_1 \cdot b_2) - r(a_1 \cdot a_2, 0)) = h(a_1 \cdot a_2, b_1 \cdot b_2) = \\
&\quad h((a_1, b_1) \cdot (a_2, b_2)) = h(x_1 \cdot x_2).
\end{aligned}
$$

This shows that h is a homomorphism and, consequently, a topological isomorphism as required. $\qquad\square$

Theorem 4.2. *A complemented subalgebra of the product of uncountable family of Fréchet algebras is topologically isomorphic to the product of Fréchet algebras. More formally, if X is a complemented subalgebra of the product $\prod\{B_t \colon t \in T\}$ of Fréchet algebras B_t, $t \in T$, then X is topologically isomorphic to the product $\prod\{F_j \colon j \in J\}$, where F_j is a complemented subalgebra of the product $\prod\{B_t \colon t \in T_j\}$ with $|T_j| = \omega$ for each $j \in J$.*

Proof. Let X be a complemented subalgebra of the uncountable product $B = \prod\{B_t \colon t \in T\}$ of Fréchet algebras B_t, $t \in T$, where T is an indexing set with $|T| = \tau > \omega$. There exists a continuous homomorphism $r \colon B \to X$ such that $r(x) = x$ for each $x \in X$. A subset $S \subseteq T$ is called r-admissible if $\pi_S(r(z)) = \pi_S(z)$ for each point $z \in \pi_S^{-1}(\pi_S(X))$.

Claim 1. *The union of an arbitrary family of r-admissible sets is r-admissible.*

Let $\{S_j \colon j \in J\}$ be a collection of r-admissible sets and $S = \bigcup\{S_j \colon j \in J\}$. Let $z \in \pi_S^{-1}(\pi_S(X))$. Clearly $z \in \pi_{S_j}^{-1}(\pi_{S_j}(X))$ for each $j \in J$ and consequently $\pi_{S_j}(r(z)) = \pi_{S_j}(z)$ for each $j \in J$. Obviously, $\pi_S(z) \in \pi_S(r(z))$ and it therefore suffices to show that the set $\pi_S(r(z))$ contains only one point. Assuming that there is a point $y \in \pi_S(r(z))$ such that $y \neq \pi_S(z)$ we conclude (remembering that $S = \bigcup\{S_j \colon j \in J\}$) that there must exist an index $j \in J$ such that $\pi_{S_j}^S(y) \neq \pi_{S_j}^S(\pi_S(z))$. But this is impossible

$$
\pi_{S_j}^S(y) \in \pi_{S_j}^S(\pi_S(r(z))) = \pi_{S_j}(r(z)) = \pi_{S_j}(z) = \pi_{S_j}^S(\pi_S(z)).
$$

Claim 2. *If $S \subseteq T$ is r-admissible, then $\pi_S(X)$ is a closed subalgebra of $B_S = \prod\{B_t \colon t \in S\}$.*

Indeed, let $i_S \colon B_S \to B$ be the canonical section of π_S (this means that $i_S = \mathrm{id}_{B_S} \triangle \mathbf{0} \colon B_S \to B_S \times B_{T-S} = B$). Consider a continuous linear map

$r_S = \pi_S \circ r \circ i_S \colon B_S \to \pi_S(X)$. Obviously, $i_S(y) \in \pi_S^{-1}(\pi_S(X))$ for any point $y \in \pi_S(X)$. Since S is r-admissible the latter implies that

$$y = \pi_S(i_S(y)) = \pi_S(r(i_S(y))) = r_S(y).$$

This shows that $\pi_S(X)$ is closed in B_S.

Claim 3. *Let S and R be r-admissible subsets of T and $S \subseteq R \subseteq T$. Then the map $\pi_S^R \colon \pi_R(X) \to \pi_S(X)$ is topologically isomorphic to the natural projection $\pi \colon \pi_S(X) \times \ker\left(\pi_S^R|\pi_R(X)\right) \to \pi_S(X)$.*

Obviously $\pi_R(X) \subseteq \pi_S(X) \times B_{R-S} \subseteq B_R = B_S \times B_{R-S}$. Consider the map $i_R = \mathrm{id}_{B_R} \triangle \mathbf{0} \colon B_R \to B_R \times B_{T-R} = B$. Also let $r_R = \pi_R \circ r \circ i_R \colon B_R \to \pi_R(X)$.

Observe that $\pi_S^R \circ r_R|(\pi_S(X) \times B_{R-S}) = \pi_S^R|(\pi_S(X) \times B_{R-S})$. Indeed, if $x \in \pi_S(X) \times B_{R-S}$, then $i_R(x) \in \pi_S^{-1}(\pi_S(X))$. Since S is r-admissible, we have $\pi_S(r(i_R(x))) = \pi_S(i_R(x))$. Consequently,

$$\pi_S^R(r_R(x)) = \pi_S^R(\pi_R(r(i_R(x)))) = \pi_S(r(i_R(x))) = \pi_S(i_R(x)) =$$
$$\pi_S^R(\pi_R(i_R(x))) = \pi_S^R(x).$$

Next observe that $r_R(x) = x$ for any point $x \in \pi_R(X)$. Indeed, since R is r-admissible and since $i_R(x) \in \pi_R^{-1}(\pi_R(X))$ we have

$$r_R(x) = \pi_R(r(i_R(x))) = \pi_R(i_R(x)) = x.$$

Application of Lemma 4.1 (with $X = \pi_R(X)$, $Y = \pi_S(X)$, $B = B_{R-S}$, $p = \pi_S^R|\pi_R(X)$ and $r = r_R$) finishes the proof of Claim 3.

Claim 4. *Every countable subset of T is contained in a countable r-admissible subset of T.*

Let A be a countable subset of T. Our goal is to find a countable r-admissible subset C such that $A \subseteq C$. By Theorem 3.3, there exist a countable subset C of T and a continuous homomorphism $r_C \colon B_C \to \mathrm{cl}_{B_C}(\pi_C(X))$ such that $A \subseteq C$ and $\pi_C \circ r = r_C \circ \pi_C$. Consider a point $y \in \pi_C(X)$. Also pick a point $x \in X$ such that $\pi_C(x) = y$. Then

$$y = \pi_C(x) = \pi_C(r(x)) = r_C(\pi_C(x)) = r_C(y).$$

This shows that $r_C|\pi_C(X) = \mathrm{id}_{\pi_C(X)}$. It also follows that $\pi_C(X)$ is closed in B_C.

In order to show that C is r-admissible let us consider a point $z \in \pi_C^{-1}(\pi_C(X))$. By the observation made above, $r_C(\pi_C(z)) = \pi_C(z)$. Finally

$$\pi_C(z) = r_C(\pi_C(z)) = \pi_C(r(x))$$

which implies that C is r-admissible.

Since $|T| = \tau$, we can write $T = \{t_\alpha \colon \alpha < \tau\}$. Since the collection of countable r-admissible subsets of T is cofinal in $\exp_\omega T$ (see Claim 4), each element $t_\alpha \in T$ is contained in a countable r-admissible subset $A_\alpha \subseteq T$. According to Claim

1, the set $T_\alpha = \bigcup\{A_\beta \colon \beta \leq \alpha\}$ is r-admissible for each $\alpha < \tau$. Consider the projective system

$$\mathcal{S} = \{X_\alpha, p_\alpha^{\alpha+1}, \tau\},$$

where

$$X_\alpha = \pi_{T_\alpha}(X) \ \text{ and } \ p_\alpha^{\alpha+1} = \pi_{T_\alpha}^{T_{\alpha+1}}|\pi_{T_{\alpha+1}}(X) \ \text{ for each } \ \alpha < \tau.$$

Since $T = \bigcup\{T_\alpha \colon \alpha < \tau\}$, it follows that $X = \operatorname{proj lim}\mathcal{S}$. Obvious transfinite induction based on Claim 3 shows that
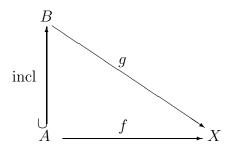
$$X = \operatorname{proj lim}\mathcal{S} = X_0 \times \prod\{\ker\left(p_\alpha^{\alpha+1}\right) \colon \alpha < \tau\}.$$

Since, by the construction, A_α is a countable r-admissible subset of T, it follows from Claim 2 that X_0 and $\ker\left(p_\alpha^{\alpha+1}\right)$, $\alpha < \tau$, are Fréchet algebras. This finishes the proof of Theorem 4.2. $\square$

5. Injective objects of the category $\mathcal{L}CS$

In this section we investigate injective objects of the category $\mathcal{L}CS$ of locally convex Hausdorff topological vector spaces and their continuous linear maps.

Recall that an object X of the category $\mathcal{L}SC$ is *injective* if any continuous linear map $f \colon A \to X$, defined on a linear subspace of a space B, admits a continuous linear extension $g \colon B \to X$ (i.e. $g|A = f$). Here is the corresponding diagram

We start with the metrizable case. The following statement is probably known (consult with [7, Theorem C.3.4], [3, pp.6–9]).

Proposition 5.1. *The following conditions are equivalent for any Banach space* X:

(1) X *is an injective object of the category* $\mathcal{L}CS$.
(2) X *is an injective object of the category* $\mathcal{B}AN$.
(3) X *is isomorphic to a complemented subspace of* $\ell_\infty(J)$ *for some set* J.

Proof. The implication (1) $\Longrightarrow$ (2) is trivial.

(2) $\Longrightarrow$ (3). The Banach space X can be identified with a closed linear subspace of the space $\ell_\infty(J)$ for some set J. By condition (2), there exists a

linear continuous map $r\colon \ell_\infty(J) \to X$ such that $r(x) = x$ for each $x \in X$. This obviously implies that X is a complemented subspace of $\ell_\infty(J)$.

$(3) \Longrightarrow (1)$. First let us show that the Banach space $\ell_\infty(J)$ (for any set J) is an injective object of the category $\mathcal{B}AN$. Let

$$f\colon X \to \ell_\infty(J) = \left\{\{x_j\colon j \in J\} \in \prod\{\mathbb{C}_j\colon j \in J\}\colon \sup\{||x_j||\colon j \in J\} < \infty\right\},$$

where $\mathbb{C}_j$, $j \in J$, stands for a copy of $\mathbb{C}$, be a continuous linear map defined on a closed linear subspace X of a Banach space Y. Since f is bounded, for each $j \in J$ we have

$$||f(x)_j|| \leq \sup\{||f(x)_j||\colon j \in J\} = ||f(x)|| \leq ||f|| \cdot ||x||, \ x \in X.$$

This shows that $||\pi_j \circ f|| \leq ||f||$, where $\pi_j\colon \ell_\infty(J) \to \mathbb{C}_j$ denotes the canonical projection onto the j-th coordinate. By the Hahn-Banach Theorem, the linear map $\pi_j \circ f\colon X \to \mathbb{C}_j$, $j \in J$, admits a continuous linear extension $g_j\colon Y \to \mathbb{C}_j$ such that $||g_j|| = ||f \circ \pi_j|| \leq ||f||$. Consequently,

$$\sup\{||g_j(y)||\colon j \in J\} \leq \sup\{||g_j|| \cdot ||y||\colon j \in J\} \leq ||f|| \cdot ||y|| < \infty, \ y \in Y.$$

This shows that the map $g\colon Y \to \ell_\infty(J)$, given by letting

$$g(y) = \{g_j(y)\colon j \in J\}, y \in Y,$$

is well defined. Obviously g is continuous, linear and extends the map f. Therefore $\ell_\infty(J)$ is indeed an injective object of the category $\mathcal{B}AN$.

Next consider a complete locally convex topological vector space Y, its closed linear subspace Z and a continuous linear map $f\colon Z \to X$, where X is a complemented subspace of the Banach space $\ell_\infty(J)$ for some set J. We need to show that f admits a continuous linear extension $g\colon Y \to X$. Identify Y with a closed linear subspace of the product $\prod\{B_t\colon t \in T\}$ of Banach spaces B_t, $t \in T$. Clearly Z is a closed linear subspace of $\prod\{B_t\colon t \in T\}$. By Lemma 3.1, there exist a finite subset T_f and a continuous linear map $f'\colon \mathrm{cl}_{B_{T_f}}\left(\pi_{T_f}(Z)\right) \to X$ such that $f = f' \circ \pi_{T_f}|Z$. Clearly $\mathrm{cl}_{B_{T_f}}\left(\pi_{T_f}(Z)\right)$ is a closed linear subspace of the Banach space $\mathrm{cl}_{B_{T_f}}\left(\pi_{T_f}(Y)\right)$. The first part of the proof of this implication, coupled with condition (3), implies that X is an injective object of the category $\mathcal{B}AN$. Therefore the map f' admits a continuous linear extension $g'\colon \mathrm{cl}_{B_{T_f}}\left(\pi_{T_f}(Y)\right) \to X$. It only remains to note that the map $g = g' \circ \pi_{T_f}|Y$ is a continuous linear extension of f. $\qquad\square$

Theorem 5.2. *The following conditions are equivalent for a locally convex topological vector space X:*

(1) X is an injective object of the category $\mathcal{L}CS$.

(2) X *is isomorphic to the product* $\prod\{F_t\colon t \in T\}$, *where each* F_t, $t \in T$, *is a complemented subspace of the product* $\prod\{\ell_\infty(J_{t_n})\colon n \in \omega\}$.

Proof. (2) $\Longrightarrow$ (1). By Proposition 5.1, $\ell_\infty(J)$ is an injective object of the category $\mathcal{L}CS$ for any set J. Obviously the product of an arbitrary collection of injective objects of the category $\mathcal{L}CS$ is also an injective object of this category. Consequently, the Fréchet space F_t, $t \in T$, as a complemented subspace of $\prod\{\ell_\infty(J_{t_n})\colon n \in \omega\}$ is injective. Finally, the space X, as a product of injectives, is an injective object of the category $\mathcal{L}CS$.

(1) $\Longrightarrow$ (2). The space X can be identified with a closed linear subspace of the product $\prod\{B_t\colon t \in T\}$ of Banach spaces B_t, $T \in T$. Each of the spaces B_t can in turn be identified with a closed linear subspace of the space $\ell_\infty(J_t)$ for some set J_t, $t \in T$. Condition (1) implies in this situation that X is a complemented subspace of the product $\prod\{\ell_\infty(J_t)\colon t \in T\}$. The required conclusion now follows from Theorem 4.2. $\qquad\square$

6. Projective objects of the category $\mathcal{L}CS$

6.1. Complemented subspaces of uncountable sums of Banach spaces.

In this section we consider coproducts in the category of complete locally convex topological vector spaces. These are called locally convex direct sums ([9, p. 55] or simply direct sums [8, p. 89]. We are mainly interested in direct sums $\bigoplus\{B_t\colon t \in T\}$ of (uncountable) collections of Banach spaces B_t, $t \in T$. Let us recall corresponding definitions. Let $\{B_t\colon t \in T\}$ be an arbitrary collection of (Banach) spaces. For each $t \in T$ the space B_t is identified (via an obvious isomorphism) with the following subspace

$$B_t = \Big\{ \{x_t\colon t \in T\} \in \prod\{B_t\colon t \in T\}\colon x_{t'} = 0 \text{ whenever } t' \neq t \Big\}$$

of the product $\prod\{B_t\colon t \in T\}$ of (Banach) spaces B_t, $t \in T$. As a set the direct sum $\bigoplus\{B_t\colon t \in T\}$ coincides with the subset of the product $\prod\{B_t\colon t \in T\}$ consisting of points with finitely many non-zero coordinates (in other words, $\bigoplus\{B_t\colon t \in T\}$ is the vector subspace of $\prod\{B_t\colon t \in T\}$ spanned by $\bigcup\{B_t\colon t \in T\}$). The topology on $\bigoplus\{B_t\colon t \in T\}$ is the finest locally convex topology for which each of the natural embeddings $B_t \hookrightarrow \bigoplus\{B_t\colon t \in T\}$ is continuous. Categorical description of this construction is also useful. Let B be a locally convex topological vector space containing each of the space B_t, $t \in T$, as a subspace. Suppose that for any choice of continuous linear maps $f_t\colon B_t \to C$, where C is a locally convex topological vector space, there exists an unique continuous linear map $f\colon B \to C$ such that $f|B_t = f_t$ for each $t \in T$. Then

B is canonically isomorphic to the direct sum $\bigoplus\{B_t\colon t \in T\}$. A straightforward verification shows that if $S \subseteq T$, then $\bigoplus\{B_t\colon t \in S\}$ can be canonically identified with the subspace of $\bigoplus\{B_t\colon t \in T\}$ consisting of points $\left\{\{x_t\colon t \in T\} \in \bigoplus\{B_t\colon t \in T\}\colon x_t = 0 \text{ for each } t \in T - S\right\}$. Note also that if $S \subseteq R \subseteq T$, then the map

$$\pi_S^R\colon \bigoplus\{B_t\colon t \in R\} \to \bigoplus\{B_t\colon t \in S\},$$

defined by letting

$$\pi_S^R\left(\{x_t\colon t \in R\}\right) = \begin{cases} x_t, & \text{if } t \in S \\ 0, & \text{if } t \in R - S, \end{cases}$$

is continuous and linear.

Proposition 6.1. *Let* $f\colon \bigoplus\{B_t\colon t \in T\} \to \bigoplus\{B_t\colon t \in T\}$ *be a continuous linear map of the direct sum of Banach spaces* B_t, $t \in T$, *into itself. Suppose that* $S \subseteq T$ *and* $f\left(\bigoplus\{B_t\colon t \in S\}\right) \subseteq \bigoplus\{B_t\colon t \in S\}$. *If* $|T| > \omega$, *then the collection*

$$\mathcal{K}_S = \left\{A \in \exp_\omega(T - S)\colon f\left(\bigoplus\{B_t\colon t \in S \cup A\}\right) \subseteq \bigoplus\{B_t\colon t \in S \cup A\}\right\}$$

is cofinal and ω-*closed in* $\exp_\omega(T - S)$.

Proof. Consider the following relation L_S on the set $(exp_\omega(T - S))^2$:

$$L_S = \Big\{(A, C) \in (exp_\omega(T - S))^2\colon A \subseteq C \text{ and }$$
$$f\left(\bigoplus\{B_t\colon t \in S \cup A\}\right) \subseteq \bigoplus\{B_t\colon t \in S \cup C\}\Big\}$$

Next we perform the spectral search with respect to L_S.

   Existence Condition. We have to show that for each $A \in exp_\omega(T - S)$ there exists $C \in exp_\omega(T - S)$ such that $(A, C) \in L_S$.

   We begin with the following observation.

   Claim. *For each* $j \in T$ *there exists a finite subset* $C_j \subseteq T$ *such that* $f(B_j) \subseteq \bigoplus\{B_t\colon t \in C_j\}$.

   The unit ball $X = \{x \in B_j\colon \|x\|_j \leq 1\}$ (here $\|\cdot\|_j$ denotes a norm of the Banach space B_j) being bounded in B_j is, by [9, Theorem 6.3], bounded in $\bigoplus\{B_t\colon t \in T\}$. Continuity of f guarantees that $f(X)$ is also bounded in $\bigoplus\{B_t\colon t \in T\}$. Applying [9, Theorem 6.3] once again, we conclude that there exists a finite subset $C_j \subseteq T$ such that $f(X) \subseteq \bigoplus\{B_t\colon t \in C_j\}$. Finally the linearity of f implies that $f(B_j) \subseteq \bigoplus\{B_t\colon t \in C_j\}$.

Let now $A \in \exp_\omega(T - S)$. For each $j \in A$, according to Claim, there exists a finite subset $C_j \subseteq T$ such that $f(B_j) \subseteq \bigoplus\{B_t \colon t \in C_j\}$. Let $\widehat{C} = \cup\{C_j \colon j \in A\}$. Observe that $|\widehat{C}| \leq \omega$. Clearly $f(B_j) \subseteq \bigoplus\{B_t \colon t \in \widehat{C}\}$ for each $j \in A$. The linearity of f guarantees in this situation that

$$f\left(\bigoplus\{B_t \colon t \in A\}\right) \subseteq \bigoplus\{B_t \colon t \in \widehat{C}\}.$$

Finally let $C = (\widehat{C} - S) \cup A$. Since, by our assumption, $f\left(\bigoplus\{B_t \colon t \in S\}\right) \subseteq \bigoplus\{B_t \colon t \in S\}$, it follows that $f\left(\bigoplus\{B_t \colon t \in S \cup A\}\right) \subseteq \bigoplus\{B_t \colon t \in S \cup C\}$. Therefore $(A, C) \in L_S$.

Majorantness Condition. Let $(A, C) \in L_S$, $D \in \exp_\omega(T - S)$ and $C \subseteq D$. Condition $(A, C) \in L_S$ implies that $f\left(\bigoplus\{B_t \colon t \in S \cup A\}\right) \subseteq \bigoplus\{B_t \colon t \in S \cup C\}$. The inclusion $C \subseteq D$ implies that $\bigoplus\{B_t \colon t \in C\} \subseteq \bigoplus\{B_t \colon t \in D\}$. Consequently

$$f\left(\bigoplus\{B_t \colon t \in S \cup A\}\right) \subseteq \bigoplus\{B_t \colon t \in S \cup C\} \subseteq \bigoplus\{B_t \colon t \in S \cup D\},$$

which means that $(A, D) \in L_S$.

ω**-closeness Condition.** Let $(A_i, C) \in L_S$, $i \in \omega$, where $\{A_i \colon i \in \omega\}$ is a countable chain in $\exp_\omega(T - S)$. We need to show that $(A, C) \in L_S$, where $A = \cup\{A_i \colon i \in \omega\}$. Condition $(A_i, C) \in L_S$ implies that $f\left(\bigoplus\{B_t \colon t \in S \cup A_i\}\right) \subseteq \bigoplus\{B_t \colon t \in S \cup C\}$, $i \in \omega$. Also observe that

$$\bigoplus\left\{\bigoplus\{B_t \colon t \in A_i\} \colon i \in \omega\right\} = \bigoplus\{B_t \colon t \in A\}.$$

Consequently, by the linearity of f,

$$f\left(\bigoplus\{B_t \colon t \in S \cup A\}\right) = f\left(\bigoplus\left\{\bigoplus\{B_t \colon t \in S \cup A_i\} \colon i \in \omega\right\}\right) \subseteq$$
$$\bigoplus\{B_t \colon t \in S \cup C\}.$$

By Proposition 2.5, the set of L_S-reflexive elements of $\exp_\omega(T - S)$ is cofinal and ω-closed in $\exp_\omega(T - S)$. It only remains to note that an element $A \in \exp_\omega(T - S)$ is L_S-reflexive if and only if $A \in \mathcal{K}_S$. $\square$

Remark 6.1. Proposition 6.1 is valid only for uncountable sums (compare with Remark 3.2). In order to see this consider an uncountable sum $\bigoplus\{B_t \colon t \in T\}$ of Banach spaces and suppose that $f \colon \bigoplus\{B_t \colon t \in T\} \to \bigoplus\{B_t \colon t \in T\}$

is a topological isomorphism. Proposition 6.1, applied to the map f, guarantees the existence of a cofinal and ω-closed subset $\mathcal{K}_\emptyset(f)$ of $\exp_\omega T$ satisfying the following property: $f\left(\bigoplus\{B_t\colon t \in A\}\right) \subseteq \left(\bigoplus\{B_t\colon t \in A\}\right)$ for each $A \in \mathcal{K}_\emptyset(f)$. The same Proposition 6.1, applied to the map f^{-1} (recall that f is a topological isomorphism), guarantees the existence of a cofinal and ω-closed subset $\mathcal{K}_\emptyset(f^{-1})$ of $\exp_\omega T$ satisfying the following property: $f^{-1}\left(\bigoplus\{B_t\colon t \in A\}\right) \subseteq \left(\bigoplus\{B_t\colon t \in A\}\right)$ for each $A \in \mathcal{K}_\emptyset(f^{-1})$. By Proposition 2.3, the intersection $\mathcal{K} = \mathcal{K}_\emptyset(f) \cap \mathcal{K}_\emptyset(f^{-1})$ is still cofinal and ω-closed in $\exp_\omega T$. Obviously, for each $A \in \mathcal{K}$ the restriction

$$f\left|\bigoplus\{B_t\colon t \in A\}\; :\; \bigoplus\{B_t\colon t \in A\} \to \bigoplus\{B_t\colon t \in A\}\right.$$

is a topological isomorphism.

Again, for countable sums such a phenomenom is impossible. In order to see this consider the inductive sequences

$$\mathcal{S}_{\mathrm{even}} = \{\mathbb{C}^{2n}, i_{2n}^{2(n+1)}, \omega\} \; \text{ and } \; \mathcal{S}_{\mathrm{odd}} = \{\mathbb{C}^{2n+1}, i_{2n+1}^{2(n+1)+1}, \omega\},$$

where

$$i_{2n}^{2(n+1)}\colon \mathbb{C}^{2n} \hookrightarrow \mathbb{C}^{2n}\bigoplus\mathbb{C}^2 \text{ and } i_{2n+1}^{2(n+1)+1}\colon \mathbb{C}^{2n+1} \hookrightarrow \mathbb{C}^{2n+1}\bigoplus\mathbb{C}^2$$

denote the natural embeddings. Clearly the limit spaces $\varinjlim \mathcal{S}_{\mathrm{even}}$ and $\varinjlim \mathcal{S}_{\mathrm{odd}}$ of these inductive systems are topologically isomorphic, but $\mathcal{S}_{\mathrm{even}}$ and $\mathcal{S}_{\mathrm{odd}}$ do not contain isomorphic cofinal subsystems.

Theorem 6.2. *A complemented subspace of the sum of uncountable family of Banach spaces is isomorphic to the sum of* (**LB**)*-spaces. More formally, if X is a complemented subspace of the sum $\bigoplus\{B_t\colon t \in T\}$ of Banach spaces B_t, $t \in T$, then X is isomorphic to the sum $\bigoplus\{F_j\colon j \in J\}$ where F_j is a complemented subspace of the countable sum $\bigoplus\{B_t\colon t \in T_j\}$ where $|T_j| = \omega$ for each $j \in J$.*

Proof. Let X be a complemented subspace of an uncountable sum $\bigoplus\{B_t\colon t \in T\}$ of Banach spaces B_t, $t \in T$. Clearly there exists a continuous linear map $r\colon \bigoplus\{B_t\colon t \in T\} \to X$ such that $r(x) = x$ for each $x \in X$. A subset $S \subseteq T$ is called r-admissible if $r\left(\bigoplus\{B_t\colon t \in S\}\right) \subseteq \bigoplus\{B_t\colon t \in S\}$.

For a subset $S \subseteq T$, let $X_S = r\left(\bigoplus\{B_t\colon t \in S\}\right)$.

We state properties of r-admissible subsets needed below.

Claim 1. *If $S \subseteq T$ is an r-admissible, then $X_S = X\bigcap\left(\bigoplus\{B_t\colon t \in S\}\right)$.*

Indeed, if $y \in X_S$, then there exists a point $x \in \bigoplus \{B_t \colon t \in S\}$ such that $r(x) = y$. Since S is r-admissible, it follows that

$$y = r(x) \in r\left(\bigoplus\{B_t \colon t \in S\}\right) \subseteq \bigoplus\{B_t \colon t \in S\}.$$

Clearly, $y \in X$. This shows that $X_S \subseteq X \bigcap \left(\bigoplus\{B_t \colon t \in S\}\right)$.

Conversely. If $y \in X \bigcap \left(\bigoplus\{B_t \colon t \in S\}\right)$, then $y \in X$ and hence, by the property of r, $y = r(y)$. Since $y \in \bigoplus\{B_t \colon t \in S\}$, it follows that $y = r(y) \in r\left(\bigoplus\{B_t \colon t \in S\}\right) = X_S$.

Claim 2. *The union of an arbitrary collection of r-admissible subsets of T is r-admissible.*

This is matter of a straightforward verification of the definition of the r-admissibility.

Claim 3. *Every countable subset of T is contained in a countable r-admissible subset of T.*

This follows from Proposition 6.1 applied to the map r.

Claim 4. *If $S \subseteq T$ is an r-admissible subset of T, then $r_S(x) = x$ for each point $x \in X_S$, where $r_S = r \Big|\left(\bigoplus\{B_t \colon t \in S\}\right) \colon \bigoplus\{B_t \colon t \in S\} \to X_S$.*

This follows from the corresponding property of the map r.

Claim 5. *If S and R are r-admissible subset of T and $S \subseteq R$, then X_R is isomorphic to the sum $X_S \bigoplus \ker\left(f_S^R\right)$, where the map $f_S^R \colon X_R \to X_S$ is defined by letting[3] $f_S^R(x) = r_S\left(\pi_S^R(x)\right)$ for each $x \in X_R$.*

Observe that $f_S^R(x) = x$ for each $x \in X_S$.

Claim 6. *If S and R are r-admissible subsets of T and $S \subseteq R$, then there exists a continuous linear map $i_S^R \colon \ker\left(f_S^R\right) \to \ker\left(\pi_S^R\right)$ such that $r_R \circ i_S^R = \mathrm{id}_{\ker\left(f_S^R\right)}$.*

Let $x \in \ker\left(f_S^R\right)$. Then $r_S\left(\pi_S^R(x)\right) = 0$ and consequently,

$$\pi_S^R\left(x - \pi_S^R(x)\right) = \pi_S^R(x) - \pi_S^R\left(\pi_S^R(x)\right) = \pi_S^R(x) - \pi_S^R(x) = 0.$$

This shows that by letting $i_S^R(x) = x - \pi_S^R(x)$ for each $x \in \ker\left(f_S^R\right)$, we indeed define a map $i_S^R \colon \ker\left(f_S^R\right) \to \ker\left(\pi_S^R\right)$. Finally observe that

$$r_S^R\left(i_S^R(x)\right) = r_R\left(x - \pi_S^R(x)\right) = r_R(x) - r_R\left(\pi_S^R(x)\right) = r_R(x) - r_S\left(\pi_S^R(x)\right) = \\ r_R(x) = x.$$

In other words, $r_S^R \circ i_S^R = \mathrm{id}_{\ker\left(f_S^R\right)}$ as required.

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[3]Recall that the definition of the map π_S^R is given right before Proposition 6.1 on page 19.

Let $|T| = \tau$. Then we can write $T = \{t_\alpha \colon \alpha < \tau\}$. Since the collection of countable r-admissible subsets of T is cofinal in $\exp_\omega T$ (see Claim 3), each element $t_\alpha \in T$ is contained in a countable r-admissible subset $A_\alpha \subseteq T$. According to Claim 2, the set $T_\alpha = \bigcup\{A_\beta \colon \beta \leq \alpha\}$ is r-admissible for each $\alpha < \tau$. Consider the inductive system

$$\mathcal{S} = \{X_\alpha, i_\alpha^{\alpha+1}, \tau\},$$

where

$$X_\alpha = X_{T_\alpha} = X \bigcap r\left(\bigoplus\{B_t \colon t \in T_\alpha\}\right) \quad \text{(see Claim 1)}$$

and

$$i_\alpha^{\alpha+1} \colon X_\alpha \to X_{\alpha+1}$$

denotes the natural inclusion for each $\alpha < \tau$. For a limit ordinal number $\beta < \tau$ the sum $\bigoplus\{B_t \colon t \in T_\beta\}$ is topologically isomorphic to the limit space of the direct system $\left\{\bigoplus\{B_t \colon t \in T_\alpha\}, j_{T_\alpha}^{T_{\alpha+1}}, \alpha < \beta\right\}$, where

$$j_{T_\alpha}^{T_{\alpha+1}} \colon \bigoplus\{B_t \colon t \in T_\alpha\} \to \bigoplus\{B_t \colon t \in T_{\alpha+1}\}$$

is the natural inclusion. This observation, coupled with Claim 4, implies that $X_\beta = \operatorname{inj lim}\{X_\alpha, i_\alpha^{\alpha+1}, \alpha < \beta\}$ for each limit ordinal number $\beta < \tau$.

In particular X is topologically isomorphic to the limit space of the inductive system $\{X_\alpha, i_\alpha^{\alpha+1}, \alpha < \tau\}$.

For each $\alpha < \tau$, according to Claim 5, the inclusion $i_\alpha^{\alpha+1} \colon X_\alpha \to X_{\alpha+}$ is topologically isomorphic to the inclusion $X_\alpha \hookrightarrow X_\alpha \bigoplus \ker\left(f_{T_\alpha}^{T_{\alpha+1}}\right)$. In this situation the straightforward transfinite induction shows that X is topologically isomorphic to the direct sum

$$X_0 \bigoplus \left(\bigoplus \left\{\ker\left(f_{T_\alpha}^{T_{\alpha+1}}\right) \colon \alpha < \tau\right\}\right).$$

Since, by construction, the set T_0 is countable, Claim 4 guarantees that X_0 is an (**LB**)-space. Similarly, since the set $T_{\alpha+1} - T_\alpha = A_{\alpha+1}$ is countable, Claim 6 guarantees that the space $\ker\left(f_{T_\alpha}^{T_{\alpha+1}}\right)$ is an (**LB**)-space for each $\alpha < \tau$. This completes the proof of Theorem 6.2. $\qquad\square$

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SASKATCHEWAN, MCLEAN HALL, 106 WIGGINS ROAD, SASKATOON, SK, S7N 5E6, CANADA

E-mail address: chigogid@math.usask.ca